Journal of Engineering Physics and Thermophysics, Vol. 72, No. 3, 1999

VIBRATIONAL PROCESSES IN MECHANICAL SYSTEMS

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UDC 531.8

A method is presented and mathematically rigorously justified for calculating vibrational processes in mechanical systems described by nonlinear differential equations. An example of its implementation is considered and it is shown that this method leads to known self-oscillation conditions.

Physical processes occurring in mechanical objects are rather complicated in the general statement of a problem, and their mathematical analogs are reduced to nonlinear differential equations that are not solvable in finite form or in quadratures. Therefore in solving such mathematical models, qualitative and approximate methods are of great importance, the role and significance of which for practical applications grow steadily. For this purpose we suggest a general method for numerical-analytical investigation of vibrational processes, highly applicable in practice in carrying out engineering calculations, in order to solve nonlinear differential equations that describe nonlinear dynamic processes in mechanical objects with ideal and nonideal energy sources and with the influence of nonlinear friction forces and useful resistance on physical processes in the models [1-4].

Suppose that the general nonlinear differential equation of a vibrational system has the form

$$u'' + f(u)\,u' + g(u) = h(t)\,. \tag{1}$$

It does not appear possible to integrate Eq. (1) in general. To eliminate this drawback, we suggest using the method of equivalent linearization for deriving simple analytical relations that are convenient in practice in performing engineering calculations, following which the functions $f(u)$ and $g(u)$ are linearized by means of the substitution

$$f(u)\,u' \to K_1(u_0)\,\tilde{u}'\,, \tag{2}$$

$$g(u) \to K_2(u_0)\,\tilde{u}\,. \tag{3}$$

Here $\tilde{u}$ denotes the approximate solution with respect to the exact u:

$$K_1(u_0) = f(u_0)\,, \tag{4}$$

$$K_2(u_0) = \frac{g(u_0)}{u_0}\,, \tag{5}$$

The condition $u_0 \neq 0$ can always be achieved by shifting the sought solution by the constant quantity $a \neq 0$; for this, we choose $u_*(t) = u(t) + a$.

The initial conditions for the exact and linearized solutions at $t = 0$ are as follows:

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 1062-0125/99/7203-0464$22.00

$$u(0) = \tilde{u}(0) = u_0 \neq 0; \tag{6}$$

$$u'(0) = \tilde{u}'(0) = u_0'. \tag{7}$$

Thus, the solution of nonlinear differential equation (1) is reduced to the solution of the ordinary differential equation

$$\tilde{u}'' + K_1(u_0)\,\tilde{u}' + K_2(u_0)\,\tilde{u} = h(t). \tag{8}$$

Integration of differential equation (8) is not difficult. We evaluate the error between the exact and approximate solutions

$$z(t) = u(t) - \tilde{u}(t). \tag{9}$$

To do this, we subtract Eq. (8) from Eq. (1). As a result, we have

$$z''(t) = Q(u, u', t), \tag{10}$$

$$Q(u, u', t) = K_1(u_0)\,\tilde{u}' + K_2(u_0)\,\tilde{u} - f(u)\,u' - g(u) = f(u)\,(u_0' - u') +$$

$$+ (f(u_0) - f(u))\,u_0' - f(u_0)\,(u_0' - \tilde{u}') - g(u) + g(u_0) + g(u_0)\,\frac{(\tilde{u} - u_0)}{u_0}. \tag{11}$$

Having integrated Eq. (10) over t twice, with allowance for the fact that $z(0) = z'(0) = 0$ we obtain

$$z(t) = \frac{1}{2}\int_0^t (t - \tau)\,Q(u, u', \tau)\,d\tau. \tag{12}$$

From Eq. (12) we have

$$z'(t) = \frac{1}{2}\int_0^t Q(u, u', \tau)\,d\tau. \tag{13}$$

Then we introduce the norm

$$\| z \|_\lambda = \max_t \left\{\exp(-\lambda t)\,(|z(t)| + |z'(t)|)\right\},\ \lambda > 0. \tag{14}$$

We evaluate it assuming that the functions $f(u, u')$, $F(u) = \int_{a_0}^{u(t)} f(\tau)d\tau$, and $g(u)$ satisfy Lipschitz conditions on the segment $[0, T]$:

$$|f(u) - f(u_0)| \leq M_1\,|u - u_0|,\ |F(u) - F(u_0)| \leq M_2\,|u - u_0|, \tag{15}$$

$$|g(u) - g(u_0)| \leq N_1\,|u - u_0|. \tag{16}$$

We consider the following differences:

$$|\tilde{u} - u_0| \leq |z| + |u - u_0|; \tag{17}$$

$$|\tilde{u}' - u_0'| \leq |z'| + |u' - u_0'|. \tag{18}$$

By virtue of inequalities (15)-(18), from Eqs. (12) and (13) we have

$$|z(t)| \leq \frac{1}{2}\int_0^t |t-\tau|\ |Q(u, u', \tau)|\ d\tau, \tag{19}$$

$$|z'(t)| \leq \frac{1}{2}\int_0^t |Q(u, u', \tau)|\ d\tau. \tag{20}$$

where

$$|Q(u, u', t)| \leq |f(u_0)|\ |z'| + \frac{|g(u_0)|}{|u_0|}|z| + \left\{M_2 + |f(u_0)|\right\} \times$$

$$\times |u' - u_0'| + \left\{\frac{|g(u_0)|}{|u_0|} + M_1 |u_0'| + N_1\right\} |u - u_0|. \tag{21}$$

We select the constants A_* and B_* from the following conditions:

$$A_* = \max_{t, u_0, u_0'} \left\{|K_1(u_0)|, \frac{|g(u_0)|}{|u_0|}\right\},$$

$$B_* = \max_{t, u_0, u_0'} \left\{|K_1(u_0)| + M_2, \frac{|g(u_0)|}{|u_0|} + M_1 |u_0'| + N_1\right\}. \tag{22}$$

Then

$$|Q(u, u', t)| \leq A_*(|z'(t)| + |z(t)|) + B_*(|u' - u_0'| + |u - u_0|). \tag{23}$$

Therefore,

$$\exp(-\lambda t)(|z'(t)| + |z(t)|) \leq \frac{1}{2}\int_0^t \exp(-\lambda(t-\tau))(1 - |t-\tau|)(A_* \exp(-\lambda\tau)(|z'(\tau)| +$$

$$+ |z(t)|) + B_* \exp(-\lambda\tau)(|u'(\tau) - u_0'| + |u(\tau) - u_0|))\, d\tau. \tag{24}$$

Based on formula (14), Eq. (24) yields

$$\| z \|_\lambda \leq \frac{1}{2}(A_* \| z \|_\lambda + B_* \| u - u_0 \|_\lambda) C(\lambda, t), \tag{25}$$

where

$$C(\lambda, t) = (\lambda^{-1} + \lambda^{-2})(1 - \exp(-\lambda t)) - T\lambda^{-1} \exp(-\lambda t).$$

The functional dependence $C(\lambda, t)$ has the horizontal asymptote $C(\lambda, t) \to C_*(\lambda)$ at $t \to \infty$, where $C_*(\lambda) = (\lambda + 1)/\lambda^2$.

From Eq. (25) it follows that the error between exact solution (1) and approximate solution (8) does not exceed the quantity

$$\| z \|_\lambda \leq \frac{B_* C(\lambda, T)}{2 - A_* C(\lambda, T)} \| u - u_0 \|_\lambda. \tag{26}$$

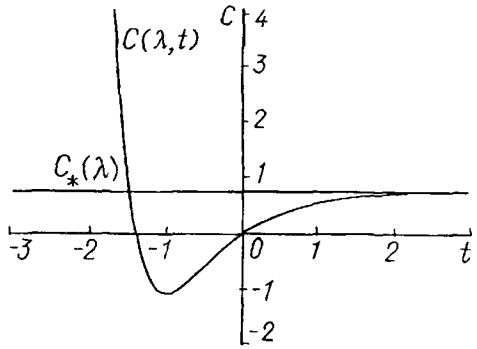

Fig. 1. Plot of $C(\lambda, t)$ and $C_*(\lambda)$ for $\lambda = 2.0$.

We fix the arbitrary parameter λ by the condition

$$C(\lambda, T) < \frac{2}{A_* + B_*}. \tag{27}$$

Taking into account that $C_*(\lambda) \cong C(\lambda, t)$ at $t \to \infty$ (see Fig. 1), the value of the parameter λ can easily be determined:

$$\lambda > \frac{1}{4}(A_* + B_* + \sqrt{(A_* + B_*)(8 + A_* + B_*)}).$$

In this case, $\| z \|_\lambda \leq \| u - u_0 \|_\lambda$.

The explicit form of Eq. (8) for approximate solution (1) allows us to write self-oscillation conditions for it in the form [2-4]

$$K_1(u_0) < 0, \quad K_2(u_0) > \frac{1}{4} K_1^2(u_0), \quad \forall\, u_0. \tag{28}$$

By virtue of Eq. (8), for the frequency of the self-oscillations we have

$$\omega = \sqrt{\left(K_2(u) - \frac{1}{4} K_1^2(u_0)\right)}. \tag{29}$$

The method suggested above for calculating vibrational processes can also be extended in an obvious way to mechanical systems simulated by equations that differ from Eq. (1). Here we will restrict ourselves to consideration of a specific example. For this purpose, we apply the procedure developed to calculatig frictional self-oscillations of the friction drive of a circular saw [1, 5-7] that are simulated by the equation

$$J\ddot{\varphi} + c\varphi = P - F(\dot{\varphi}). \tag{30}$$

Here J is the constant moment of inertia; c is the stiffness coefficient of the elastic element; P is the constant magnitude of the external load; $F(\dot{\varphi})$ is the characteristic of the friction force, described by the nonlinear function of the angular velocity

$$F(\dot{\varphi}) = F_0 - F_1\dot{\varphi} + F_2\dot{\varphi}^3 - F_3\dot{\varphi}^5,$$

where $F_0 > 0$, $F_1 > 0$, $F_2 > 0$, $F_3 > 0$ are constant quantities [1, 5, 6].

Restricting ourselves to three terms for $F(\dot{\varphi})$, we rewrite Eq. (30) in the form

$$\ddot{\tilde{\varphi}} + \frac{1}{J}(-F_1 + F_2\dot{\tilde{\varphi}}^2)\dot{\tilde{\varphi}} = \frac{c}{J}\tilde{\varphi} = \frac{P}{J} - \frac{F_0}{J}. \tag{31}$$

Repeating the above reasoning, we linearize Eq. (31):

$$\ddot{\tilde{\varphi}} + K_1(\dot{\varphi}_0)\dot{\tilde{\varphi}} + K_2\tilde{\varphi} = \frac{P}{J} - \frac{F_0}{J}, \tag{32}$$

where

$$K_1(\dot{\varphi}_0) = \frac{1}{J}(-F_1 + F_2\dot{\varphi}_0^2), \quad K_2 = \frac{c}{J}. \tag{33}$$

Equation (32) simulates rather well the mechanical processes of frictional self-oscillations of the friction drive of a circular saw and makes it possible to write the self-oscillation conditions [3, 4]

$$F_1 > F_2\dot{\varphi}_0^2, \quad c > \frac{1}{4}\frac{1}{J}(F_1 - F_2\dot{\varphi}_0^2)^2, \quad \forall\, \dot{\varphi}_0.$$

With allowance for notation (33) it is seen that the procedure developed with the three-term approximation of $F(\dot{\varphi})$ leads to well-known results obtained by other methods and enables one to justify their reliability mathematically rigorously [1, 5-7].

NOTATION

u, unknown generalized coordinate; u', u'', first and second derivatives of the generalized coordinate; $f(u)$, $g(u)$, and $h(t)$, known functions of the generalized coordinate and time t; M_1, M_2, N_1, Lipschitz constants.

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